Lecture 27.

Last Time:

Suppose that X and Y are continuous, independent random variables with densities $f_X(x)$ and $f_Y(y)$ respectively. Then the random variable $X+Y$ has a density function given by the convolution of f_X and f_Y:

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,dy = \int_{-\infty}^{\infty} f_Y(z-x) f_X(x)\,dx$$

Ex: Sum of two uniform RV's on $(0,1)$. X, Y indep, uniform on $(0,1)$, $Z = X+Y$.

Then
$$f_X = f_Y = \begin{cases} 1 & \text{on } (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

So
$$f_Z = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,dy$$

$$= \int_0^1 f_X(z-y)\,dy$$

$0 < z-y < 1$.
so $\underbrace{0 < z < 1+y}$
$0 < y < 1 \Rightarrow$
$\boxed{0 < z < 2.}$

Now,
$$f_X(z-y) = \begin{cases} 1 & \text{for } 0 < z-y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

So if $0 < z \le 1$ (so $0 < z-y < 1$ iff $0 < y < z$)

$$f_Z(z) = \int^z du = z.$$

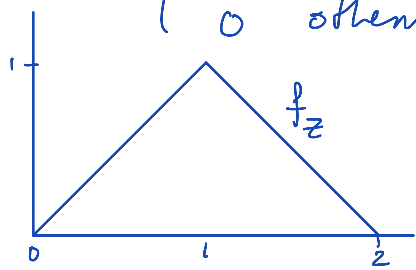
If $1 < z < 2$, (so $0 < z - y < 1$ iff $z-1 < y <$)

then $f_Z(z) = \int_{z-1}^{1} dy = 1 - (z-1) = 2 - z.$

$z - y < 1$

so $y > z - 1$

So $f_Z(z) = \begin{cases} z & 0 < z \leq 1 \\ 2-z & 1 < z < 2 \\ 0 & \text{otherwise.} \end{cases}$



Sums of Independent Gamma Random variables

Recall that X is a gamma random variable with parameters (λ, t) if

$$f_X(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{t-1}}{\Gamma(t)} & 0 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Functions of this form with fixed λ are closed under convolution. Thus:

Proposition: Suppose X and Y are Γ RVs with params (s, λ) and (t, λ), resp. Then $X+Y$ is Γ with params $(s+t, \lambda)$.

Pf: Using convolutions, we have

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy = \int_{0}^{z} f_X(z-y) f_Y(y)\, dy$$

since $f_X(z-y) = 0$ for $y \geq z$

since $f_Y(y)=0$ for $y<0$

$$= \frac{1}{\Gamma(s)\Gamma(t)} \int_0^z \lambda e^{-\lambda(z-y)} (\lambda(z-y))^{s-1} \lambda e^{-\lambda y} (\lambda y)^{t-1} dy$$

$$= \frac{\lambda\lambda^{s-1}\lambda\lambda^{t-1}}{\Gamma(s)\Gamma(t)} \int_0^z e^{-\lambda z} \cancel{e^{\lambda y}} (z-y)^{s-1} \cancel{e^{-\lambda y}} y^{t-1} dy$$

$$= \frac{\lambda^s \lambda^t e^{-\lambda z}}{\Gamma(s)\Gamma(t)} \int_0^z (z-y)^{s-1} y^{t-1} dy$$

Let $y = zx$

$z\,dx = dy$

$$= \underbrace{\frac{\lambda^s \lambda^t e^{-\lambda z} z^{s+t-1}}{\Gamma(s)\Gamma(t)}}_{\text{constant}} \underbrace{\int_0^1 (1-x)^{s-1} x^{t-1} dx}_{\text{constant}}$$

$$= K e^{-\lambda z} z^{s+t-1}.$$

Since $1 = \int_0^\infty K e^{-\lambda z} z^{s+t-1} dz$ set

$$\frac{1}{K} = \int_0^\infty e^{-\lambda z} z^{s+t-1} dz \qquad z = \frac{w}{\lambda} \qquad dz = \frac{dw}{\lambda}.$$

$$= \int_0^\infty e^{-w} \lambda^{s+t-1} w^{s+t-1} \lambda dw = \frac{1}{\lambda}\int_0^\infty e^{-w} w^{s+t-1} dw$$

$$= \frac{\Gamma(s+t)}{\lambda}$$

Hence $f_{X+Y}(z) = \dfrac{\lambda e^{-\lambda z} z^{s+t-1}}{\Gamma(s+t)}$ Hence $X+Y$ is a Γ RV with params $(s+t, \lambda)$. $\square$.

Corollary: Let X_i, $1 \le i \le n$, be Γ with parameters (t_i, λ). Then $\sum_{i=1}^n X_i$ is a Γ RV with parameters $(\Sigma t_i, \lambda)$.

Pf: Induction on n. The previous prop. is the base case.

Have a nice day! :)

Sums of Independent Normal Random variables

The following is sometimes called the "reproductive property of normal Random variables":

Proposition: Let $X_i = N(\mu_i, \sigma_i^2)$, $i = 1, \ldots, n$ be independent normal random variables. Then

$$\sum X_i = N\left(\sum \mu_i, \sum \sigma_i^2\right).$$

Proof: We break the proof up into several steps.

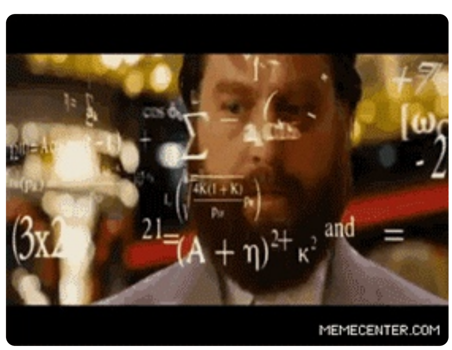
First, let $X = N(0, \sigma^2)$ and $Y = N(0,1)$.

Set $c = \frac{1}{2\sigma^2} + \frac{1}{2} = \frac{1+\sigma^2}{2\sigma^2}$.

Then,

$$f_X(a-y) f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(a-y)^2}{2\sigma^2}\right] \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-y^2}{2}\right]$$

$$= \vdots$$



...The magic of algebra...

$$= \frac{1}{2\pi\sigma} \exp\left[\frac{-a^2}{2(1+\sigma^2)}\right] \exp\left[-c^2\left(y - \frac{a}{1+\sigma^2}\right)^2\right]$$

Therefore:

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\,dy = \frac{1}{2\pi\sigma} \exp\left(\frac{-a^2}{2(1+\sigma^2)}\right) \int_{-\infty}^{\infty} \exp(-cx^2)\,dx$$

$$= C \exp\left(\frac{-a^2}{2(1+\sigma^2)}\right)$$, C some constant not depending on a.

We integrate to find C. $\Rightarrow$ $\boxed{X+Y = N(0,\ 1+\sigma^2).}$

Next step: Suppose $X_i = N(\mu_i, \sigma_i^2)$, $i = 1, 2$.

Then

$$X_1 + X_2 = \sigma_2 \left(\underbrace{\frac{X_1 - \mu_1}{\sigma_2}}_{= N\left(0, \frac{\sigma_1^2}{\sigma_2^2}\right)} + \underbrace{\frac{X_2 - \mu_2}{\sigma_2}}_{= N(0,1)} \right) + \mu_1 + \mu_2.$$

(multiply by 1 subtract 0)

By the previous step,

$$\frac{X_1 - \mu_1}{\sigma_2} + \frac{X_2 - \mu_2}{\sigma_2} = N\left(0,\ 1 + \frac{\sigma_1^2}{\sigma_2^2}\right)$$

$$= N\left(0,\ \frac{\sigma_1^2 + \sigma_2^2}{\sigma_2^2}\right).$$

Now, recall that $\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$ for constants $a, b \in \mathbb{R}$, hence.

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}\left(\sigma_2 N\left(0, \frac{\sigma_1^2 + \sigma_2^2}{\sigma_2^2}\right) + \mu_1 + \mu_2\right)$$

$$= \sigma_2^2 \left(\frac{\sigma_1^2 + \sigma_2^2}{\sigma_2^2}\right)$$

$$\left(\sigma_2^2 \right)$$

$$= \sigma_1^2 + \sigma_2^2 .$$

$$E[X_1 + X_2] = \sigma_2 \left(E\left[N\left(0, \tfrac{\sigma_1^2}{\sigma_2^2}\right)\right] + E[N(0,1)] \right) + \mu_1 + \mu_2$$

$$= \sigma_2 (0 + 0) + \mu_1 + \mu_2 = \mu_1 + \mu_2 .$$

Hence $X_1 + X_2 = N(\mu_1 + \mu_2 , \sigma_1^2 + \sigma_2^2)$.

The general case follows immediately by induction on n, with $X_1 + X_2$ as the base case.